Algorithms and moduli spaces for differential equations

Berkenbosch, Maint

Chapter 4

The Riemann-Hilbert problem and examples

In this chapter we will give some examples of moduli spaces of differential equations, and we describe the connection with the Riemann-Hilbert problem.

4.1 The classical Riemann-Hilbert problem

We start by briefly recalling the classical Riemann-Hilbert problem as described in [PS03] Chapter 6.

Let (M, ∇) be a regular singular connection over $\mathbb{C}(z)$ with singular locus equal to $S = \{s_1, \cdots, s_r\} \subset \mathbb{P}^1_{\mathbb{C}}$. This means that M is a finite dimensional $\mathbb{C}(z)$-vector space, and $\nabla : M \to \mathbb{C}(z)dz \otimes M$ is a regular singular connection (see [PS03] 6.4.2). One defines a monodromy map associated to (M, ∇) in the following manner. Write $V := \ker(\mathbb{C}((z-b)) \otimes_{\mathbb{C}(z)} M, \nabla)$ for the local solution space at a regular point $b \in \mathbb{P}^1_{\mathbb{C}} \setminus S$ and define $\pi_1 := \pi_1(\mathbb{P}^1_{\mathbb{C}} \setminus S, b)$. Let $\lambda \in \pi_1$ be a loop, then we can make an analytic continuation of the local solutions V along λ. This analytic continuation defines a linear map on V, so we can associate an element of $\mathrm{GL}(V)$ to λ. This process results in a map $\pi_1 \to \mathrm{GL}(V)$, called the *monodromy map*. The question if for any given representation $\rho : \pi_1 \to \mathrm{GL}(V)$ there is a regular singular connection with

Theorem 6.22 of [PS03]). For general (M, ∇) this is not always the case.

We will conclude by briefly describing how a connection $(\mathcal{M}, \nabla)$ on $\mathbb{P}^1_{\mathbb{C}}$ with a prescribed monodromy representation can be constructed. We will use a generalization of this construction in the next section.

Write $U := \mathbb{P}^1_{\mathbb{C}} \setminus S$, so $\pi_1 := \pi_1(U, 0)$. We start by constructing a regular connection on U with the prescribed monodromy. For this consider the universal covering $u : \widetilde{U} \to U$ of U. Define a connection $(\mathcal{N}, \nabla)$ on $\widetilde{U}$ by $\mathcal{N} := \mathbb{C}^n \otimes \mathcal{O}_{\widetilde{U}}$, and $\nabla(v \otimes f) = v \otimes f'$ for all $v \in \mathbb{C}^n,\ f \in \mathcal{O}_{\widetilde{U}}$. Furthermore we define a π_1-action on $\mathcal{N}$ by $\lambda(v \otimes f) = \rho(\lambda)(v) \otimes (f \circ \lambda^{-1})\ \forall\ \lambda \in \pi_1$. The vector bundle $\mathcal{N}$ corresponds to the geometric vector bundle $\mathbb{C}^n \times \widetilde{U}$, and the corresponding π_1-action is given by $\lambda(v, \tilde{u}) = (\rho(\lambda)(v), \lambda(\tilde{u})),\ v \in \mathbb{C}^n,\ \tilde{u} \in \widetilde{U}$. Indeed in this way we get for a section $h \times id : \widetilde{U} \to \mathbb{C}^n \times \widetilde{U},\ h \in \mathcal{N}(\widetilde{U})$, that $(\lambda(h) \times id)(\tilde{u}) = \lambda(h \times id(\lambda^{-1}(\tilde{u})))$. It is clear that the quotient $\pi_1 \setminus (\mathbb{C}^n \times \widetilde{U})$ defines a geometric vector bundle on $U = \pi_1 \setminus \widetilde{U}$, with corresponding vector bundle $\mathcal{M}_U := \mathcal{N}^{\pi_1}$. The π_1-action on $\mathcal{N}$ commutes with ∇. So we find an induced connection ∇_U on $\mathcal{M}_U$. Write $\mathcal{L} := \ker(\nabla_U, \mathcal{M}_U)$. The only thing left to show is that $\mathcal{L}$ is the local system corresponding to ρ. There is a one to one correspondence between local systems on U and (trivial) local systems on $\widetilde{U}$ with a π_1-action. Under this correspondence $\mathcal{L}$ clearly corresponds to $\mathbb{C}^n$ with the defined π_1-action which is given by ρ. This proves that $(\mathcal{M}_U, \nabla_U)$ has monodromy given by ρ.

We now want to extend this connection $(\mathcal{M}_U, \nabla_U)$ to a connection on $\mathbb{P}^1_{\mathbb{C}}$. Let $s \in S$, and consider the pointed disk $U_s^* := 0 < |z - s| < \varepsilon$. We will construct a connection on $U_s := |z - s| < \varepsilon$ that glues to the restriction of $(\mathcal{M}_U, \nabla_U)$ to U_s^*. For this we consider the local solution space V_s at $s + \frac{\varepsilon}{2}$. The circle around s through $s + \frac{\varepsilon}{2}$ induces a monodromy map $B \in \mathrm{GL}(V_s)$. Choose $A \in \mathrm{End}(V)$ such that $e^{2\pi i A} = B$, then we define the connection ∇_s on the vector bundle $\mathcal{M}_s := \mathcal{O}|_{U_s} \otimes V_s$ by $\nabla_s(f \otimes v) = df \otimes v + z^{-1} \otimes A(v)$. The restriction of $(\mathcal{M}_s, \nabla_s)$ to U_s^* clearly has local monodromy B. By [PS03] 6.6-3 this restriction is isomorphic to the restriction of $(\mathcal{M}_U, \nabla_U)$ to U_s^*. Therefore we can glue the connection $(\mathcal{M}_s, \nabla_s)$ to $(\mathcal{M}_U, \nabla_U)$. In this way we obtain the desired connection $(\mathcal{M}, \nabla)$ on $\mathbb{P}^1_{\mathbb{C}}$ extending $(\mathcal{M}_U, \nabla_U)$.

4.2 The Riemann-Hilbert problem for families

We will now consider the Riemann-Hilbert problem for families of differential equations. Let Y be an analytic manifold, and let $S := \{s_1, \cdots, s_r\}$ be a set of points in $\mathbb{P}^1_{\mathbb{C}} \setminus \{0, \infty\}$. Suppose that $(\mathcal{M}, \nabla)$ is an analytic family of differential equations on $\mathbb{P}^1$, parametrized by Y (the definition of an analytic family is a straightforward variation of Definition 3.13). We suppose that S is the set of singular points of ∇; more precisely, for every $y \in Y$ the set of singular points of $\nabla(y)$ is S.

We write $pr_1 : Y \times \mathbb{P}^1 \to Y$, $pr_2 : Y \times \mathbb{P}^1 \to \mathbb{P}^1$ for the two projection maps. Let $U := \mathbb{P}^1 \setminus S$, and $\pi_1 := \pi_1(U, 0)$. We will also write pr_1, pr_2 for the restrictions to $Y \times U$ of the two projection maps. The kernel $\mathcal{L} := \ker(\nabla)$ of $\nabla|_{Y \times U}$ is a locally free $pr_1^*(\mathcal{O}_Y)$-module of rank n, where $n = \dim(V)$. For any $a \in U$ the embedding $j_a : Y \cong Y \times \{a\} \hookrightarrow Y \times U$ defines a vector bundle $\mathcal{L}_a := j_a^*(\mathcal{L})$ on Y .

We will now define a monodromy map $\pi_1 \to \mathrm{Aut}(\mathcal{L}_0)$. Let $\lambda : [0, 1] \to U$ be a path in U. Then $(id \times \lambda)^*(\mathcal{L})$ is a $pr^*(\mathcal{O}_Y)$-module on $Y \times [0, 1]$, where $pr : Y \times [0, 1] \to Y$ is the projection map. Since $(id \times \lambda)^*(\mathcal{L})$ is a locally free sheaf, we find a canonical isomorphism $\mathcal{L}_{\lambda(0)} \xrightarrow{\sim} \mathcal{L}_{\lambda(1)}$. In particular, a closed path λ with $\lambda(0) = 0 \in \mathbb{P}^1_{\mathbb{C}}$ yields an automorphism of $\mathcal{L}_0$ and this defines a homomorphism $\pi_1 \to \mathrm{Aut}(\mathcal{L}_0)$.

Definition 4.2 We call the map $\pi_1 \to \mathrm{Aut}(\mathcal{L}_0)$ constructed above, the monodromy map associated to $(\mathcal{M}, \nabla)$. •

We now present a converse construction, which we interpret as a solution to the Riemann-Hilbert problem for families.

Theorem 4.3 *Let Y be an analytic irreducible reduced manifold with a vector bundle $\mathcal{L}$ on it. Suppose we are given a set $S := \{s_1, \cdots, s_r\} \subset \mathbb{P}^1_{\mathbb{C}}$ and a representation $\rho : \pi_1(\mathbb{P}^1_{\mathbb{C}} \setminus S) \to \mathrm{Aut}(\mathcal{L})$ satisfying the following properties.*

- *Let $\lambda_i \in \pi_1$ be loops around the points s_i with $\prod_{i=1}^{s} \lambda_i = 1$. For every $y \in Y$ we have that $\rho(\lambda_i)(y) \sim e^{2\pi i A_i}$ for some fixed $A_i \in \mathrm{M}_n(\mathbb{C})$. Here*

$\rho(\lambda_i)(y)$ denotes the automorphism on $\mathcal{L}_y/(m_y\mathcal{L}_y) \cong \mathbb{C}^n$ induced by $\rho(\lambda_i)$.

- *None of the differences of the eigenvalues of A_i is in $\mathbb{Z} \setminus \{0\}$.*

Then there exists an analytic connection $(\mathcal{M}, \nabla)$ on $Y \times \mathbb{P}^1$, with singular points in S and monodromy map given by ρ.

Proof. We can cover Y by Stein-manifolds Y_i such that $\mathcal{L}|_{Y_i}$ is free for all i. We will now construct a solution to the Riemann-Hilbert problem for families over Y_i. From the construction it can be seen that the connections on the Y_i glue to a connection on Y with the appropriate monodromy map, hence this also solves the Riemann-Hilbert problem for families over Y. From now on we assume $\mathcal{L}$ to be free and Y to be a Stein-manifold, and ρ is given as a homomorphism $\rho : \pi_1 \to \mathrm{GL}_n(\mathcal{O}(Y))$.

Let $\widetilde{U}$ be the universal covering of U. We can identify π_1 with $\mathrm{Aut}(\widetilde{U}/U)$. Write $pr_1 : Y \times \widetilde{U} \to Y$, $pr_2 : Y \times \widetilde{U} \to \widetilde{U}$ for the two projection maps. The vector bundle $\mathcal{N} := \mathcal{O}^n_{Y\times\widetilde{U}}$ can be written as $pr_1^{-1}(\mathcal{O}^n_Y)\widetilde{\otimes} pr_2^{-1}(\mathcal{O}_{\widetilde{U}})$, where $\widetilde{\otimes}$ is an "analytic tensor product", as defined in [GR71] p.179.

Remark 4.4 We note that $pr_1^{-1}(\mathcal{O}_Y)$ are the analytic functions on $Y \times \widetilde{U}$ which are constant with respect to $\widetilde{U}$. The sheaf $pr_1^{-1}(\mathcal{O}_Y) \otimes_{\mathbb{C}} pr_2^{-1}(\mathcal{O}_{\widetilde{U}})$ (the usual tensor product over $\mathbb{C}$) consists of functions of the form $\sum_{i=1}^m f_i \cdot g_i$, where the f_i are constant with respect to $\widetilde{U}$ and the g_i are constant with respect to Y. Therefore the sheaf $pr_1^{-1}(\mathcal{O}^n_Y) \otimes_{\mathbb{C}} pr_2^{-1}(\mathcal{O}_{\widetilde{U}})$ consisting of n-tuples of such functions is much smaller then $\mathcal{N}$. •

We will now define a connection $(\mathcal{M}_U, \nabla_U)$ on $Y \times U$ by a construction similar to the one in the previous section. Define a π_1-action on $\mathcal{N}$ given by the formula $\lambda(v\widetilde{\otimes}f) = \rho(\lambda)v\widetilde{\otimes}(f \circ \lambda^{-1})$ for $v \in pr_1^{-1}(\mathcal{O}^n_Y)$, $f \in pr_2^{-1}(\mathcal{O}_{\widetilde{U}})$, and $\lambda \in \pi_1 \cong \mathrm{Aut}(\widetilde{U}/U)$. Let $\mathcal{M}_U := \mathcal{N}^{\pi_1}$, then $\mathcal{M}_U$ defines a vector bundle on $Y \times U$. Let $\nabla : \mathcal{N} \to \mathcal{N} \otimes \Omega_{(Y\times\widetilde{U})/Y}$ be given by $\nabla(v \otimes f) = v \otimes df$. The connection ∇ commutes with the π_1-action, and we get an induced connection $(\mathcal{M}_U, \nabla_U)$ on $Y \times U$ with monodromy representation given by ρ.

Now we will extend $(\mathcal{M}_U, \nabla_U)$ to a connection on $Y \times \mathbb{P}^1$. Let $s \in S$, and let $O_s^* := \{z \in U | 0 < |z - s| < \varepsilon\}$ be a small neighborhood of a. The inverse image of O_s^* under the natural map $u : \widetilde{U} \to U$ consists of a number of connected components. Let $\widetilde{O}_s$ be one of them, then $u : \widetilde{O}_s \to O_s^*$ is a universal covering ([F77] Section 31.4). Let $\lambda \in \pi_1$ be a loop around s. The subgroup of π_1 mapping $\widetilde{O}_s$ to itself is cyclic with generator λ.

Lemma 4.5 *Let Y be an irreducible Stein manifold over $\mathbb{C}$ and $A \in \mathrm{M}_n(\mathbb{C})$ a matrix with the property that the differences of the eigenvalues of A are not in $\mathbb{Z} \setminus \{0\}$. If $M \in \mathrm{GL}_n(\mathcal{O}(Y))$ satisfies $M(y) \sim e^{2\pi i A}$ $\forall\ y \in Y$, then there exists $B \in \mathrm{M}_n(\mathcal{O}(Y))$ with $M = e^{2\pi i B}$ and $B(y) \sim A$ $\forall\ y \in Y$.*

Proof. Let $K := \mathrm{Frac}(\mathcal{O}(Y))$, and let $\mu_1, \cdots, \mu_p$ be the distinct eigenvalues of A. Write $\nu_j := e^{2\pi i \mu_j}$, then $\nu_1, \cdots, \nu_p$ are the distinct eigenvalues of M. We can make a decomposition $M = M_{ss} M_u$, with M_{ss} semi-simple and M_u unipotent. One can write M_{ss} and M_{ss}^{-1} as polynomials in M with coefficients in $\mathbb{C}$, so $M_{ss}, M_u \in \mathrm{GL}_n(\mathcal{O}(Y))$. Let $V_i := \ker(M_{ss} - \nu_i I, K^n)$, then $K^n = V_1 \oplus \cdots \oplus V_p$. For $w \in \mathcal{O}(Y)^n$ we can write $w = w_1 + \cdots + w_p$, with $w_i \in V_i$. Now $M_{ss}^m(w) = \nu_1^m w_1 + \cdots + \nu_p^m w_p \in \mathcal{O}(Y)^n, m \geq 0$. Using the fact that the Vandermonde matrix $\begin{pmatrix} 1 & \nu_1 & \cdots & \nu_1^p \\ \vdots & & & \vdots \\ 1 & \nu_p & \cdots & \nu_p^p \end{pmatrix}$ is invertible, we see that all $w_i \in \mathcal{O}(Y)^n$, so we can write $\mathcal{O}(Y)^n = \oplus W_i,\ W_i := \ker(M_{ss} - \nu_i I, \mathcal{O}(Y)^n)$.

Let $B_{ss} \in \mathrm{M}_n(\mathcal{O}(Y))$ be the linear map that acts as multiplication by μ_i on W_i, and let B_n be defined as the finite sum $\frac{1}{2\pi i} \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j} (M_u - I)^j$. We will show that $B := B_{ss} + B_n$ satisfies the lemma. We have that $e^{2\pi i B_{ss}} = M_{ss}$ and $e^{2\pi i B_n} = M_u$ Since B_{ss} and B_n commute it is clear that $M = e^{2\pi i B}$. Furthermore $e^{2\pi i B(y)} \sim e^{2\pi i A}$ $\forall\ y \in Y$, and the eigenvalues of $B(y)$ and A correspond. By construction we have $B(y) \sim A$ $\forall\ y \in Y$. □

We find that we can write $\rho(\lambda_i) = e^{2\pi i B_i},\ B_i \in \mathrm{M}_n(\mathcal{O}(Y))$, with $B_i(y) \sim A_i$ for all $y \in Y$. Let $s = s_i$ be the singular point we fixed, then we write B for B_i.

For notational convenience we replace the covering $u : \widetilde{O}_s \to O_s^*$ by the covering $exp : \mathbb{C} \to \mathbb{C}^*,\ z \mapsto e^{2\pi i z}$. The group $\mathrm{Aut}(\mathbb{C}/\mathbb{C}^*)$ is generated by $t : z \mapsto z + 1$. The restriction of $\mathcal{N}$ to $Y \times \mathbb{C}$ is $pr_1^{-1}(\mathcal{O}_Y^n) \widetilde{\otimes} pr_2^{-1}(\mathcal{O}_{\mathbb{C}})$, and we want to calculate $(\mathcal{N}^{\langle t\rangle}, \nabla)$ explicitly. So we have to calculate the action of $t \in \mathrm{Aut}(\mathbb{C}/\mathbb{C}^*)$ on $\mathcal{N}|_{Y\times\mathbb{C}}$. Let $v(y,z)$ be a section of $\mathcal{N}|_{Y\times\mathbb{C}}$. Using the explicit description of the π_1-action on $\mathcal{N}$ given in the beginning of this construction we find $t(v(y,z)) = e^{2\pi i B}v(y,z-1)$. Write $v(y,z) = e^{2\pi i Bz}w(y,z)$, then the condition $t(v) = v$ is equivalent to $w(y,z) = w(y,z-1)$. So $t(v) = v \iff w(y,z) = \tilde{w}(y, e^{2\pi i z})$ for some section $\tilde{w}$ of $\mathcal{O}^n_{Y\times\mathbb{C}}$.

We find that $(\mathcal{N}|_{Y\times\mathbb{C}})^{\langle t\rangle} \cong \mathcal{O}^n_{Y\times\mathbb{C}^*}$ is a free vector bundle on $Y\times\mathbb{C}^*$ with generators $\{f_1,\cdots,f_n\}$, $f_i = e^{2\pi i Bz}e_i$, where $\{e_1,\cdots,e_n\}$ is the standard free basis for $\mathcal{O}^n_{Y\times\mathbb{C}^*}$. Furthermore ∇ is given by $\nabla(f_i) = 2\pi i B f_i dz$. We have that $u := e^{2\pi i z}$ is a parameter on $\mathbb{C}^*$, and we find $\nabla(f_i) = Bf_i\frac{du}{u}$. Using this formula, we can extend the connection $((\mathcal{N}|_{Y\times\mathbb{C}})^{\langle t\rangle}, \nabla)$ on $\mathbb{C}^*$ to a connection on $\mathbb{C}\supset\mathbb{C}^*$. In this way we can make an extension of $(\mathcal{M}_U, \nabla_U)$ to a connection on $Y\times\mathbb{P}^1$. □

In the following we want to construct a family of differential equations parametrized by a certain space of monodromy representations. Suppose we are given regular singular moduli problem in the sense of Chapter 2, with data $(V, \{s_1,\cdots,s_r\}, \{\frac{d}{dt_j}+\frac{C_j}{t_j}\}_{j=1}^r)$, $C_j\in\mathrm{GL}(V)$. Consider the set of corresponding monodromy representations $\mathbf{M} := \{\rho\in\mathrm{Repr}(\pi_1,V)|\rho(\lambda_j)\sim e^{-2\pi i C_j}\}$. We can identify $\mathbf{M}$ with the set $\{(M_1,\cdots,M_r)|M_j\sim e^{-2\pi i C_j}, \prod_{j=1}^r M_j = I\}$ by identifying ρ with $\{\rho(\lambda_1),\cdots,\rho(\lambda_r)\}$.

Lemma 4.6 *The set $\mathbf{M}$ is a Zariski constructible subset of $\mathrm{GL}(V)^r$ and, if all matrices C_i are diagonalizable, even Zariski closed. Furthermore the subset of $\mathbf{M}$ consisting of irreducible representations is also Zariski constructible.*

Proof. It clearly is sufficient to prove corresponding statements for the set $\mathbf{M}'$ obtained by dropping the condition $\prod_{i=1}^r M_i = I$. In proving the first statement, we may suppose $r=1$. For a diagonal matrix $C\in\mathrm{GL}(V)$ with characteristic polynomial $P_C = \prod(T-\mu_i)^{m_i}$, the set $\{ACA^{-1}|A\in\mathrm{GL}(V)\}$ is given by $\{B\in\mathrm{GL}(V)|P_B = P_C, \mathrm{rank}(B-\mu_i I) = n-m_i\ \forall\ i\}$. The latter condition is equivalent to the condition that the determinant of all $l\times l$-submatrices of $B-\mu_i I$, with $l > n-m_i$, is zero. This clearly defines a closed set. For an arbitrary matrix $C\in\mathrm{GL}(V)$, we have that B is similar to C if and only if $P_B = P_C$, and $\mathrm{rank}((B-\mu I)^m) = \mathrm{rank}((C-\mu I)^m), m = 1,\cdots,n$, for every eigenvalue μ. This defines a constructible set. To be precise,

$\mathrm{rank}(A) \leq m$ defines a closed subset of $\mathrm{GL}(V)$, so $\mathrm{rank}(A) = m$ defines a constructible subset.

We will now prove the second statement. Note that the set of matrices in $\mathrm{GL}(V)$ that leave a line $\mathbb{C} \cdot v,\ v \in V$ invariant is given by $\{M | Mv \wedge v = 0\}$, where $\wedge$ denotes the exterior product. So the set of tuples $(M_1, \cdots, M_s)$ that leave a line invariant is obtained by first taking the kernel of the map $V \setminus \{0\} \times \mathrm{GL}(V)^s \to \mathbb{C}^s,\ (v, M_1, \cdots, M_s) \mapsto (M_1 v \wedge v, \cdots, M_s v \wedge v)$ and then taking the projection of this kernel onto $\mathrm{GL}(V)^s$. This clearly defines a constructible set. The matrices that leave a subspace of dimension $l < n$ invariant, are the matrices that leave a decomposable line in $\bigwedge^l V$ invariant. This also defines a constructible set. Since the complement of a constructible set is constructible, this proves the lemma. $\square$

The family $\mathbf{M}$ of representations gives rise to a family of differential equations parametrized by $\mathbf{M}$, according to Theorem 4.3. In more detail, given $\lambda \in \pi_1$, a representation $m \in \mathbf{M}$ yields an element $m(\lambda) \in \mathrm{GL}(V)$. This defines a morphism $\rho(\lambda) : \mathbf{M} \to \mathrm{GL}(V)$ which we regard as an element $\rho(\lambda) \in \mathrm{GL}(\mathcal{O}(\mathbf{M}) \otimes V)$. We obtain a representation $\rho : \pi_1 \to \mathrm{GL}(\mathcal{O}(\mathbf{M}) \otimes V)$. By Theorem 4.3 the representation ρ gives rise to a family of differential equations $(\mathcal{M}, \nabla, V, \frac{d}{dt_i} + \frac{C_i}{t_i})$ parametrized by $\mathbf{M}$. For $m \in \mathbf{M}$, the monodromy representation of $(\mathcal{M}(m), \nabla(m))$ is clearly congruent to m. By the classical Riemann-Hilbert theorem, and Lemma 4.1 the connection $(\mathcal{M}(m), \nabla(m))$ is unique up to isomorphism.

We conclude this section by a lemma on the local invertibility of the exponential map. It states that under more general conditions than in Lemma 4.5 one can still locally construct a logarithm.

Lemma 4.7 *The map $E : M_d(\mathbb{C}) \to \mathrm{GL}_d(\mathbb{C}), A \mapsto e^{2\pi i A}$ is locally invertible in A if and only if $\lambda_i - \lambda_j \notin \mathbb{Z} \setminus \{0\}$ for all couples of eigenvalues λ_i, λ_j of A.*

Proof. We start by proving that if there are two eigenvalues λ_1, λ_2 of A with $\lambda_i - \lambda_j \in \mathbb{Z} \setminus \{0\}$, then E is not locally invertible. Write $A = SJS^{-1}$, with J in Jordan normal form. We will show that there exists a matrix $B \neq 0$, with $E(J + \varepsilon B) = E(J), \varepsilon^2 = 0$ (where we use the extension of E to a map on $M_n(\mathbb{C}[\varepsilon])$). If $E(J + \varepsilon B) = E(J)$ then also $E(A + \varepsilon SBS^{-1}) = E(A)$ holds. We can suppose that J has only two eigenvalues $\lambda, \lambda + m, m \in \mathbb{Z} \setminus \{0\}$ and only two Jordan blocks of size j and $d-j$ respectively. Subtracting $\lambda \cdot Id$ from J, we may assume that J has eigenvalues $0, m$. Define B by $B_{j+1,j}=1$, and zeros everywhere else. Then $JB = mB$, $BJ = 0$. It follows that

$$E(J+\varepsilon B)-E(J) = (\sum_{n\geq 1}\frac{(2\pi i)^n}{n!}\sum_{p=0}^{n-1} J^p B J^{n-1-p})\varepsilon = (\sum_{n\geq 1}\frac{(2\pi i)^n}{n!} m^{n-1} B)\varepsilon =$$

$$\frac{1}{m}(e^{2\pi i m}-1)B\varepsilon = 0.$$

To proof the converse, again write $A = SJS^{-1}$. We will first consider the case where J is a diagonal matrix. For a matrix B with only one nonzero entry $B_{ij}=1$, we have that $E(J+\varepsilon B)-E(J)$ also has (at most) one nonzero entry at the same place. The fact that the remaining coefficient is nonzero follows from an explicit calculation in the case $J = \begin{pmatrix} 0 & 0 \\ 0 & \lambda \end{pmatrix}$, $\lambda \notin \mathbb{Z}$. We conclude that the derivative of E at A is bijective. For the general case, write $J = D+N$, where D is diagonal, N is nilpotent, and $ND = DN$. We will use the matrix norm $\|A\| = \max\{|A_{ij}|, 1 \leq i,j \leq d\}$, which has the property $\|AB\| \leq d\|A\|\|B\|$. The idea of the proof is as follows. Local invertibility at J is equivalent to local invertibility at a conjugate SJS^{-1}. We can pick S such that $\|SNS^{-1}\|$ becomes arbitrary small. An estimate then shows that local invertibility at SDS^{-1} implies local invertibility at SJS^{-1}.

We have

$$E(J+\varepsilon B)-E(J) = (\sum_{n\geq 1}\frac{(2\pi i)^n}{n!}\sum_{p=0}^{n-1} J^p B J^{n-1-p})\varepsilon =$$

$$\varepsilon\sum_{n\geq 1}\frac{(2\pi i)^n}{n!}\sum_{p=0}^{n-1}(J^p - D^p)BJ^{n-1-p} + D^p B(J^{n-1-p} - D^{n-1-p}+$$

$$\varepsilon\sum_{n\geq 1}\frac{(2\pi i)^n}{n!}\sum_{p=0}^{n-1} D^p B D^{n-1-p}.$$

Write $a := \|\sum_{n\geq 1}\frac{(2\pi i)^n}{n!}\sum_{p=0}^{n-1} D^p B D^{n-1-p}\|$. Then $a > 0$ by the argument above. We can make the following estimate:

$$\|\sum_{n\geq 1}\frac{(2\pi i)^n}{n!}\sum_{p=0}^{n-1} J^p B J^{n-1-p}\| \geq$$

$$a - \|\sum_{n\geq 1}\frac{(2\pi i)^n}{n!}\sum_{p=0}^{n-1}(J^p - D^p)BJ^{n-1-p} + D^pB(J^{n-1-p} - D^{n-1-p})\|.$$

Write $\delta := \|N\|, s := \|D\|, t := s + \delta$. We will now use the estimate

$$\|J^p - D^p\| = \|N\sum_{k=1}^{p}\binom{p}{k}D^{p-k}N^{k-1}\| \leq d^p\delta\sum_{k=1}^{p}\binom{p}{k}s^{p-k}\delta^{k-1} \leq d^ppt^{p-1}\delta.$$

Writing $b := \|B\|$, we find that

$$\|\sum_{n\geq 1}\frac{(2\pi i)^n}{n!}\sum_{p=0}^{n-1}J^pBJ^{n-1-p}\| \geq$$

$$a - \sum_{n\geq 1}\frac{(2\pi i)^n}{n!}\sum_{p=0}^{n-1}d^{n-1}(pt^{p-1}bt^{n-1-p}\delta + (n-1-p)t^pbt^{n-p-2}\delta) =$$

$$a - \sum_{n\geq 1}\frac{(2\pi i)^n}{n!}\sum_{p=0}^{n-1}d^{n-1}(n-1)t^{n-2}b\delta = a - (\sum_{n\geq 1}\frac{(2\pi i)^n}{n!}n(n-1)(dt)^{n-2})db\delta =$$

$$a - (2\pi i)^2\sum_{n\geq 0}\frac{(2\pi i)^n}{n!}(dt)^n)db\delta = a - (2\pi i)^2 e^{2\pi i dt}db\varepsilon.$$

So for any matrix B, we can pick a basis (and therefore a small δ), such that $\|E(J + \varepsilon B) - E(J)\| > 0$, which shows that E is locally invertible at J, and therefore at A. $\square$

For a vector bundle $\mathcal{M}$ obtained by Theorem 4.3, there can be points $y \in Y$ such that the induced vector bundle $\mathcal{M}(y)$ on $\mathbb{P}^1_{\mathbb{C}}$ is not free. This situation already appears in the Lamé example as we will see later on. Before we get to the Lamé example, we will first study connections on non-free vector bundles in detail.

4.3 Connections on non-free vector bundles

We will now give a precise description of connections on non-free vector bundles, and construct a fine moduli space for such connections.

Let $\mathcal{M} \cong \mathcal{O}(a_1) \oplus \cdots \oplus \mathcal{O}(a_n)$, $a_1 \geq \cdots \geq a_n$ be a vector bundle, and $D = \sum_{i=1}^r k_i[s_i]$ a divisor of degree $k := \sum_{l=1}^r k_l$, with all $s_i \neq \infty$. We can write $\mathcal{O}(a_i)(U_0) = \mathbb{C}[z]e_i$, $\mathcal{O}(a_i)(U_\infty) = \mathbb{C}[z^{-1}]f_i$, with $f_i = z^{a_i}e_i$. A connection $\nabla : \mathcal{M} \to \Omega(D) \otimes \mathcal{M}$ is given by two connections on the free vector bundles $\mathcal{M}(U_0), \mathcal{M}(U_\infty)$, say $\nabla_0 : \mathcal{M}(U_0) \to \Omega(D)(U_0) \otimes \mathcal{M}(U_0)$ and $\nabla_\infty : \mathcal{M}(U_\infty) \to \Omega(D)(U_\infty) \otimes \mathcal{M}(U_\infty)$ that glue on $U_0 \cap U_\infty$. We have that $\Omega(D)(U_0) = \mathbb{C}[z]\frac{dz}{\prod_{l=1}^r t_l^{k_l}}$ (where as always $t_l = z - s_l$), so ∇_0 is given by a $C[z]$-linear map A on $C[z]\langle e_1, \cdots, e_n\rangle$, taking $\nabla_0(e_i) = A(e_i)\frac{dz}{\prod_{l=1}^r t_l^{k_l}}$. We wil also write A for the matrix of A on the basis $\{e_1, \cdots, e_n\}$. In the same way the connection ∇_∞ is defined by $\nabla_\infty(f_i) = B(f_i)\frac{dz}{\prod_{l=1}^r t_l^{k_l}}$, with B given by a matrix $B \in M_n(z^{k-2}\mathbb{C}[z^{-1}])$. For the connections ∇_0 and ∇_∞ to glue, we must have $\nabla_0(z^{a_i}e_i) = \nabla_\infty(f_i)$. This translates into $\prod_{l=1}^r t_l^{k_l} a_i z^{a_i-1} + z^{a_i}A_{ii} = z_i^a B_{ii}$ for $i = 1, \cdots, n$ and $z^{a_j}A_{ij} = z^{a_i}B_{ij}$ for $i, j = 1 \cdots n,\ i \neq j$. From this we obtain the following properties for A:

- $\deg(A_{ij}) \leq k + a_i - a_j - 2$ for $i \neq j$,
- $\deg(A_{ii}) = k - 1$,
- A_{ii} has as highest order coefficient $-a_i$.

Conversely, a matrix $A \in \mathrm{M}_n(\mathbb{C}[z])$ satisfying these properties defines a connection on $\mathcal{M}$.

In the following we will use the group of automorphisms of $\mathcal{M}$, so we give an explicit description of it. An automorphism ψ of $\mathcal{M}$ is given by a $C[z]$-linear automorphism of $\mathcal{M}(U_0)$ and a $C[z^{-1}]$-linear automorphism of $\mathcal{M}(U_\infty)$ that glue. So $\psi(U_0)$ is given on the basis $\{e_1, \cdots, e_n\}$ by a matrix $A \in \mathrm{GL}_n(C[z])$. Furthermore $\psi(U_\infty)$ is given on the basis $\{f_1, \cdots, f_n\}$ by a matrix $B \in \mathrm{GL}_n(C[z^{-1}])$ with $B = Z^{-1}AZ$, where Z is the diagonal matrix with $Z_{ii} = z^{a_i}$. Let $a_{n_1}, \cdots, a_{n_p}$ be the subsequence of $a_1, \cdots, a_n$ consisting of a_1 and the a_i with $a_i - a_{i-1} < 0$. Then we can write A in block form

$$A = \begin{pmatrix} A_{11} & \cdots & A_{1p} \\ 0 & \ddots & \vdots \\ \vdots & \ddots & A_{pp} \end{pmatrix}.$$

Here the $A_{ii} \in \mathrm{GL}_{n_{i+1}-n_i}(C)$ (where we take $n_{p+1} = n+1$) and the coefficients of $A_{ij},\ i > j$ are polynomials of degree $\leq a_{n_i} - a_{n_j}$. Conversely any such matrix A defines an automorphism of $\mathcal{M}$.

4.3.1 Moduli spaces of non-free connections

We will define a moduli space of connections on a vector bundle of some fixed type associated to a data set $(V, \{s_1, \cdots, s_r\}, \{\frac{d}{dz} + B_i\}_{i=1}^r)$ as in Chapter 2. We fix an ordered basis for V, say $\{e_1, \cdots, e_n\}$. Define a vector bundle $\mathcal{M}$ of type $(a_1, \cdots, a_n),\ a_1 \geq \cdots \geq a_n$ by $\mathcal{M}(U_0) = \mathbb{C}[z] \otimes V$, $\mathcal{M}(U_\infty) = \mathbb{C}[z^{-1}] \otimes (\oplus \mathbb{C} z^{a_i} e_i)$. We fix a type $(a_1, \cdots, a_n)$ with $a_1 - a_n \leq r-1$, and we will only consider connections on the corresponding vector bundle $\mathcal{M}$. Note that in case $\mathcal{M}$ has rank 2, and there exists an irreducible connection on $\mathcal{M}$, then by [PS03] Proposition 6.21 we get $a_2 - a_1 \leq r - 2$.

We start by defining a functor $\mathcal{F}^+$ in a similar way to the definition of $\mathcal{F}$ in Chapter 2, but now we do not divide out equivalence.

Definition 4.8 The functor $\mathcal{F}^+ : \{\mathbb{C}\text{-algebras}\} \to \{\text{sets}\}$ is defined as follows. For any $\mathbb{C}$-algebra R, the set $\mathcal{F}^+(R)$ consists of the tuples $(A, \{\phi_i\}_{i=1}^r)$, where:

- $A \in M_n(R[z])$ satisfies $\deg(A_{ij}) \leq k + a_i - a_j - 2$ for $i \neq j$ and $\deg(A_{ii}) = k - 1$. Furthermore A_{ii} has as highest order coefficient $-a_i$.
- the $\phi_i = \sum_{j=0}^{\infty} \phi_i(j)(t_i)^j$, $\phi_i(j) \in M_n(R)$ are automorphisms of $R[[t_i]]^n$.
- $\phi_i(\frac{d}{dz} + \frac{A}{\prod_{l=1}^r t_l^{k_l}})\phi_i^{-1} = \frac{d}{dz} + B_i,\ i = 1, \cdots, s$, where we see $\frac{A}{\prod_{l=1}^r t_l^{k_l}}$ and ϕ_i as elements of $\mathrm{End}(R[[t_i]][t_i^{-1}]^n)$. This condition can be restated as $\phi_i' = \phi_i \frac{A}{\prod_{l=1}^r t_l^{k_l}} - B_i\phi_i$. •

This functor $\mathcal{F}^+$ is represented by a $\mathbb{C}$-algebra of finite type U, as can be shown in a way similar to the proof of Theorem 2.9. We can also consider $\mathcal{F}^+$ as a contravariant functor on schemes of finite type over $\mathbb{C}$. In this setting $\mathcal{F}^+$ is represented by $\mathbb{M} := \mathrm{Spec}(U)$.

We say that two tuples $(A_1, \{\phi_i^1\}), (A_2, \{\phi_i^2\}) \in \mathcal{F}^+(R)$ are equivalent if there exists an automorphism ψ of $\mathcal{M}\otimes R$ such that $\frac{d}{dz}+\frac{A_2}{\prod_{l=1}^{r} t_l^{k_l}} = \psi^{-1}(\frac{d}{dz}+\frac{A_1}{\prod_{l=1}^{r} t_l^{k_l}})\psi$ and $\phi_i^2 = \phi_i^1 \circ \psi,\ i = 1, \cdots, r$ where we consider ψ as an element of $\mathrm{GL}_n(R[z])$ and $\mathrm{GL}_n(R[[t_i]])$ respectively. We define a functor $\mathcal{F}$ by $\mathcal{F}(R) = \mathcal{F}^+(R)/\sim$.

Theorem 4.9 *There is a coarse moduli scheme for the functor $\mathcal{F}$ defined above, which is in fact a quasi projective variety.*

Proof. Consider the group $G := \mathrm{Aut}(\mathcal{M})$. this group acts on $\mathbb{M}(\mathbb{C})$ and we want to make a quotient. We can make an embedding $G \subset \mathrm{GL}_n(\mathbb{C}[z])$. From the description of G above we see that the degree of the coefficients of elements of G is bounded by $\max_{i=1\cdots n-1}(a_i - a_{i+1})$. By our assumption on $\mathcal{M}$ this bound is less or equal to $r-2$. Therefore the map $\psi : G \to \mathrm{GL}_n(\mathbb{C})^r$ given by $A(z) \mapsto (A(s_1), \cdots, A(s_r))$ is injective. In this way we can consider G as a linear algebraic subgroup of $\mathrm{GL}_n(\mathbb{C})^r$. By [Br69] Theorem 6.8, the quotient $\mathrm{GL}_n(\mathbb{C})^r/G$ exists and is given by $(Q,\pi),\ \pi : \mathrm{GL}_n(\mathbb{C})^r \to Q$, with Q a quasi-projective variety. Let $\phi : \mathbb{M} \to \mathrm{GL}_n(\mathbb{C})^r,\ (A, \{\phi_i\}) \mapsto (\phi_1(0), \cdots, \phi_r(0))$ be the projection map. We want to use the following proposition to prove that a geometric quotient $\mathbb{M}/G$ exists and is quasi-projective.

Proposition 4.10 (Proposition 7.1 of [MFK94])
Let G be a group scheme, flat and of finite type over S. Let X and Y be schemes of finite type over S, let σ and τ be actions of G on X and Y, and let $\phi : X \to Y$ be a G-linear morphism. Assume that Y is a principal fibre bundle over an S-scheme Q, with group G, and with projection $\pi : Y \to Q$. Assume that there exists an $L \in Pic^G(X)$ which is relatively ample for ϕ, and that Q is quasi-projective over S. Then there is a scheme P, quasi-projective over S, and an S-morphism $\omega : X \to P$ such that X becomes a principal fibre bundle over P with group G, and projection ω.

This needs some explanation. A principal fibre bundle is defined as follows: let $\sigma : G \times_S X \to X$ be an action, with a geometric quotient (Q,π), then X is a principal fibre bundle over Q with group G if

- π is a flat morphism of finite type,
- the map $(\sigma, pr_2) : G \times_S X \xrightarrow{\sim} X \times_Q X \subset X \times_S X$ is an isomorphism.

By Proposition 0.9 of [MFK94] for a free action of an algebraic group G on an algebraic scheme X with geometric quotient (Q, π), the scheme X always is a principal fibre bundle over Q with group G.

We further remark that $Pic^G(X)$ is the group of G-linearized line bundles on X. For details see [MFK94].

We want to apply this proposition with $S = \mathrm{Spec}(\mathbb{C})$, $X = \mathbb{M}$, $Y = \mathrm{GL}_n^r$, and G, ϕ, Q, π as above. There are a number of conditions to be checked.

(1) ϕ is G-linear.

(2) GL_n^r is a principal fibre bundle over Q with group $\mathrm{Aut}(M)$.

(3) There exists an L as in the proposition.

Condition (1) is clearly fulfilled. For the line bundle L in (3) we can take the trivial line bundle since $\mathbb{M}$ is affine. By Proposition 0.9 of [MFK94] for a free action of an algebraic group G on an algebraic scheme Y with geometric quotient (Q, π), the scheme Y always is a principal fibre bundle over Q with group G. So to prove (3) it suffices to show that the action of G on GL_n^r is free, and that Q is a geometric quotient. The action being free means that $(\sigma, pr_2) : G \times \mathrm{GL}_n^r \to \mathrm{GL}_n^r \times_Q \mathrm{GL}_n^r$ is a closed immersion, which is the case. The fact that (Q, π) is a geometric quotient follows from the definition of a quotient in [Br69].

We will now proof that P is a coarse moduli scheme for $\mathcal{F}$ by an argument as in the proof of Proposition 5.4 of [MFK94]. There is a natural isomorphism $\phi^+ : \mathcal{F}^+ \to \mathrm{Hom}(*, P)$, which induces a natural isomorphism $\phi : \mathcal{F} \to \mathrm{Hom}(*, P)$. For (P, ϕ) to be a coarse moduli space, the folowing conditions have to be verified.

- for every algebraically closed field k, the map

  $$\phi(\mathrm{Spec}\ k) : \mathcal{F}(\mathrm{Spec}\ k) \to \mathrm{Hom}(\mathrm{Spec}\ k, P)$$

  is an isomorphism.

- given a scheme N and a morphism $\psi : \mathcal{F} \to \mathrm{Hom}(*, N)$, there is a unique morphism $\chi : \mathrm{Hom}(*, P) \to \mathrm{Hom}(*, N)$, such that $\psi = \chi \circ \phi$.

The first condition is verified since (P, ω) is a geometric quotient. To proof that the second condition is verified, consider the element $\overline{id} \in \mathcal{F}(\mathbb{M})$ induced by $id \in \mathcal{F}^+(\mathbb{M}) \cong \mathrm{Hom}(\mathbb{M}, \mathbb{M})$. To a morphism $\psi : \mathcal{F} \to \mathrm{Hom}(*, N)$, we associate the morphism $f := \psi_{\mathbb{M}}(\overline{id}) : \mathbb{M} \to N$. This induces a bijection of the set of morphisms from $\mathcal{F}$ to representable functors and the set of morphisms $f : \mathbb{M} \to N$ with N a scheme. It follows that the second condition is verified, and therefore (P, ϕ) is a coarse moduli space. $\square$

4.4 The Lamé equation

We will now consider the moduli problem with data

$$(\{s_1, \cdots, s_4\}, \{\frac{d}{dt_i} + \frac{1}{t_i}\begin{pmatrix} \frac{1}{4} & 0 \\ 0 & -\frac{1}{4} \end{pmatrix}\}_{i=1}^4).$$

The corresponding set of "monodromy representations" $\mathbf{M}$ defined above is given by $\mathbf{M} = \{(M_1, \cdots, M_4) | M_i \sim \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \prod_{i=1}^4 M_i = 1\}$. Write $M_i := \begin{pmatrix} p_i & q_i \\ r_i & -p_i \end{pmatrix}$, $i = 1, 2, 3$, then he coordinate ring of $\mathbf{M}$ is given by $R := \mathbb{C}[p_1, q_1, r_1, p_2, q_2, r_2, p_3, q_3, r_3]/I$,

$$I = \langle p_1^2 + q_1 r_1 + 1, p_2^2 + q_2 r_2 + 1, p_3^2 + q_3 r_3 + 1, f \rangle$$

$$f := -p_3 q_2 r_1 + p_2 q_3 r_1 + p_3 q_1 r_2 - p_1 q_3 r_2 - p_2 q_1 r_3 + p_1 q_2 r_3.$$

The following properties of $\mathbf{M}$ are known (but also easily verified).

- $\mathbf{M}$ is a five dimensional variety.
- The group $\mathrm{PGL}_2(\mathbb{C})$ acts on $\mathbf{M}$ (by conjugation) and on its coordinate ring R. The ring $R^{\mathrm{PGL}_2} = \mathbb{C}[t_1, t_2, t_3]/\langle t_1^2 + t_2^2 + t_3^2 + t_1 t_2 t_3 - 4 \rangle$ is the ring of invariants, where $t_1 := \mathrm{Tr}(M_2 M_3)$, $t_2 := \mathrm{Tr}(M_1 M_3)$, $t_3 := \mathrm{Tr}(M_1 M_2)$. This follows immediately from [Bo03] Section 2.
- The variety $\mathbf{M}^{\mathrm{PGL}_2} := \mathrm{Spec}(R^{\mathrm{PGL}_2})$ has 4 singular points, namely $(t_1, t_2, t_3) \in \{(-2, 2, 2), (2, -2, 2), (2, 2, -2), (-2, -2, -2)\}$. Each one of these points corresponds to multiple $\mathrm{PGL}_2(\mathbb{C})$-orbits. After deleting the 4 singular points and their preimages in $\mathbf{M}$ we obtain a good quotient under $\mathrm{PGL}_2(\mathbb{C})$. In particular $\mathbf{M}$ is reduced and irreducible.

- The preimage of the 4 singular points of $\mathbf{M}^{\mathrm{PGL}_2}$ in $\mathbf{M}$ precisely consists of all reducible representations in $\mathbf{M}$.
- The complement $\mathbf{M}_{\mathrm{ir}}$ is a smooth connected variety.

By Theorem 4.3 we obtain a family of differential equations parametrized by $\mathbf{M}$, say $(\mathcal{M}, \nabla, \mathbb{C}^2,$ local data $)$. For every irreducible representation $m \in \mathbf{M}$, the following lemma shows that $\mathcal{M}(m)$ is either free, or of type $(1, -1)$.

Lemma 4.11 *Let $(\mathcal{M}, \nabla)$ be an irreducible connection of rank 2 on $\mathbb{P}^1_{\mathbb{C}}$ with four singular points such that the sum of the local exponents at each singular point is 0. Then the vector bundle $\mathcal{M}$ is of the type $\mathcal{O}(a) \oplus \mathcal{O}(-a), a \in \{0, 1\}$.*

Proof. Because the sum of the local exponents of $(\mathcal{M}, \nabla)$ is zero at each singular point, the induced connection $\bigwedge^2 \nabla$ on $\bigwedge^2 \mathcal{M}$ is everywhere regular. Since $\mathbb{P}^1_{\mathbb{C}}$ is simply connected, $\bigwedge^2 \mathcal{M}$ is the trivial line bundle, and $\bigwedge^2 \nabla$ is the trivial connection. So $\mathcal{M}$ is of the type $\mathcal{O}(a) \oplus \mathcal{O}(-a), a \geq 0$. By [PS03] Proposition 6.21, the defect of $\mathcal{M}$ is ≤ 2. This proves the lemma. □

We will now show that the set $\mathbf{M}^{(1,-1)} := \{m \in \mathbf{M}_{\mathrm{ir}} | \mathcal{M}(m) \cong \mathcal{O}(1) \oplus \mathcal{O}(-1)\}$ is nonempty.

Proposition 4.12 $\mathbf{M}^{(1,-1)}$ *is a non-trivial divisor in* $\mathbf{M}_{\mathrm{ir}}$.

Proof. By Remark 3.12 (4) we have that $\mathbf{M}^{(1,-1)}$ is a divisor. So we only need to show that $\mathbf{M}^{(1,-1)}$ is non-empty. As we saw in Section 4.2 the connection $(\mathcal{M}(m), \nabla(m))$ is uniquely determined for every $m \in \mathbf{M}$. Therefore we only have to construct a connection on $\mathcal{O}(1) \oplus \mathcal{O}(-1)$ with the correct local behavior. By the description of connections on non-free vector bundles above, we get that such a connection is given by a matrix A of the form

$$A = \begin{pmatrix} a_0 + a_1 z + a_2 z^2 - z^3 & b_0 + b_1 z + b_2 z^2 + b_3 z^3 + b_4 z^4 \\ c_0 & d_0 + d_1 z + d_2 z^2 + z^3 \end{pmatrix}.$$

We want that the connection is locally at the points s_i formally isomorphic to $\frac{d}{dz} + \frac{1}{t_i}\begin{pmatrix} \frac{1}{4} & 0 \\ 0 & -\frac{1}{4} \end{pmatrix}$, so the Laurent series expansion of $\frac{A}{\prod_{l=1}^{r} t_l^{k_l}}$ at a point s_i has to be of the form $\frac{A_i}{t_i}$ + h.o.t., with A_i similar to $\begin{pmatrix} \frac{1}{4} & 0 \\ 0 & -\frac{1}{4} \end{pmatrix}$. The

A_i are of the form $\begin{pmatrix} p_i & q_i \\ r_i & -p_i \end{pmatrix}$, $p_i^2 + q_i r_i = \frac{1}{16}$ for all i, and we can write $\frac{A}{\prod_{l=1}^r t_l^{k_l}} = \sum_{i=1}^4 \frac{A_i}{t_i} + \begin{pmatrix} 0 & b_4 \\ 0 & 0 \end{pmatrix}$. We find that $p_1 + p_2 + p_3 + p_4 + 1 = 0$, and r_2, r_3, r_4 are multiples of r_1. So we get a 5-dimensional family of tuples $(A_1, \cdots, A_4, b_4)$, and hence a 5-dimensional family X of connections on $\mathcal{O}(1) \oplus \mathcal{O}(-1)$.

The automorphism group G of $\mathcal{O}(1) \oplus \mathcal{O}(-1)$ is $\{\begin{pmatrix} a & b + cz + dz^2 \\ 0 & e \end{pmatrix},\ a, e \neq 0\}$. A connection given by a matrix A is equivalent to the one given by the matrix $\tilde{A} = \Phi^{-1}\Phi' + \Phi^{-1}A\Phi$, with $\Phi \in G$. We can construct a one dimensional subfamily of X consisting of matrices of the form $\begin{pmatrix} -z^3 & * \\ 1 & z^3 \end{pmatrix}$ parameterized by b_4 with no equivalent elements. In case $s_i = i,\ i = 1, \cdots, 4$ this family is

$$\{\begin{pmatrix} -z^3 & \frac{6265}{4} - 3015z + 1800z^2 - 350z^3 + b_4(24 - 50z + 35z^2 - 10z^3 + z^4) \\ 1 & z^3 \end{pmatrix},\ b_4 \in \mathbb{C}\}.$$

□

We remark that the above does not imply that $\mathbf{M}^{(1,-1)}$ is 1-dimensional. Indeed, let $\mathcal{M}$ be the vector bundle on $\mathbb{P}^1_{\mathbf{M}}$ given by the Riemann-Hilbert construction. The type of $\mathcal{M}(m),\ m \in \mathbf{M}$ is $(0,0)$ or $(1,-1)$, and since $\mathbf{M}$ is irreducible we find by Remarks 3.12 that $\mathbf{M}^{(1,-1)}$ is an analytic divisor on $\mathbf{M}$ (since $\mathbf{M}^{(1,-1)}$ is nonempty). We find that $\mathbf{M}^{(1,-1)}$ is 4-dimensional, and so there are isomorphic connections in $\mathbf{M}^{(1,-1)}$.

We can make similar calculations (which are in fact simpler) for the case of a free vector bundle. We get a 5-dimensional space of connections, with an action of the group SL_2. There exists a categorical quotient, which maps all reducible connections to four points. This 2-dimensional quotient is actually a geometric quotient on the space of irreducible connections.

monodromy representation equivalent to ρ is known as the weak Riemann-Hilbert problem. This question has a positive answer, which is precisely formulated in [PS03], Theorem 6.15.

The strong Riemann-Hilbert problem asks whether for a given representation $\rho : \pi_1 \to \mathrm{GL}(V)$ there is a *Fuchsian* connection over $\mathbb{C}(z)$ with monodromy representation equivalent to ρ. A Fuchsian connection is a connection which for the differentiation $\frac{d}{dz}$ can be written in the form $\frac{d}{dz} + \sum_{i=1}^{r} \frac{A_i}{z-s_i}$, with $A_i \in M_n(\mathbb{C})$. In general, for a given ρ there is no such Fuchsian connection. However under some conditions on ρ the strong Riemann-Hilbert problem has a positive answer, see sections 6.4 and 6.5 of [PS03] for details.

The strong Riemann-Hilbert problem can be restated in terms of connections on vector bundles. For a connection $(\mathcal{M}, \nabla)$ on $\mathbb{P}^1_{\mathbb{C}}$ (where $\mathcal{M}$ is not necessarily free), we get an induced connection (M_η, ∇_η) over $\mathbb{C}(z)$ by localization at the generic fibre. Therefore we can associate a monodromy map to $(\mathcal{M}, \nabla)$. It is easily seen that the strong Riemann-Hilbert problem precisely asks whether there is a connection on a *free* vector bundle with some given monodromy map.

For a representation $\rho : \pi_1 \to \mathrm{GL}(V)$, by [PS03] Theorem 6.15, we find an associated connection (M, ∇) over $\mathbb{C}(z)$. The following lemma states how we can associate a connection over $\mathbb{P}^1_{\mathbb{C}}$ to (M, ∇).

Lemma 4.1 (Lemma 6.18 of [PS03]) *Let (M, ∇) be a regular singular connection over $\mathbb{C}(z)$ with singular locus S. For every $s \in S$ we choose a local parameter t_s. For every $s \in S$ let $\Lambda_s \subset \widehat{M_s} := \mathbb{C}((t_s)) \otimes M$ be a lattice that satisfies $\nabla(\Lambda_s) \subset \frac{dt_s}{t_s} \otimes \Lambda_s$ (the existence of such a lattice is equivalent to (M, ∇) being regular singular at s). Then there is a unique regular singular connection $(\mathcal{M}, \nabla)$ on $\mathbb{P}^1_{\mathbb{C}}$ with singular locus in S such that:*

1. *For every open $U \subset \mathbb{P}^1_{\mathbb{C}}$, one has $\mathcal{M}(U) \subset M$.*

2. *The generic fibre of $(\mathcal{M}, \nabla)$ is (M, ∇).*

3. *$\widehat{\mathcal{M}}_s = \Lambda_s$ for all $s \in S$.*

In the case where (M, ∇) is irreducible, one can make a choice for the lattices Λ_s in such a way that the corresponding vector bundle $\mathcal{M}$ is free (see